Implications of Unchecked Exponential Growth[1]

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Abstract

Treatments of the exponential function in Calculus textbooks are often cursory in that they treat unchecked (Malthusian) growth processes without examination of their consequences. Through the use of problems posed to students, we illustrate the implications of such processes when applied to growth of populations and to growth in the rates of consumption of nonrenewable resources.

I. INTRODUCTION

The study of the natural exponential function e^x and its application to various growth and decay phenomena have become standard features of courses in Calculus. Although textbooks typically include applications to the growth of populations and to continuously compounded interest accounts, such treatments are often perfunctory and do not display the ultimate outcomes of unchecked exponential growth [1]. A more thorough analysis is well within student capabilities and we present here several problems to extend the students' understanding of the effects and consequences of such growth. In particular we examine applications to population growth and to growth in the consumption rate of a nonrenewable resource such as oil. These examples were inspired by the work of the late Albert A. Bartlett [2],

II. POPULATION GROWTH

The instantaneous size of a population undergoing Malthusian growth is given by

$$N(t) = N_0\, e^{kt} \qquad (1)$$

in which N_0 is the population size at time $t = 0$ and k is a positive constant. A characteristic measure of the rate of growth is the doubling time, T_2, given by $kT_2 = \mathrm{Ln}\ 2$.

Although these concepts are readily understood, the consequences of exponential population growth are not easily appreciated. Hence It is instructive to pose questions involving computation of T_2 for human populations and we outline two such exercises below Each assumes growth of the form of (1) under conditions of the current World growth rate of 1.1%/yr, corresponding to $T_2 \approx 63$ years.

Question (1): In how many doubling times will the density of human beings on Earth approach one person per square meter of solid land surface? This requires solution for n of the equation $2^n\rho = 1$ in which ρ represents the current population density of $\sim 5 \times 10^{-5}\ \mathrm{m}^{-2}$ .There results $n \approx 14$ doubling times, corresponding to 880 years. Students invariably remark that they would feel crowded long before this state is attained.

Question (2): In how many doubling times will the mass of human beings on Earth equal the mass of the Earth itself? This requires solution for n of $2^n\, m = M$ in which m represents the present mass of human beings on Earth and M is the Earth mass of $\approx 6 \times 10^{24}$ kg. Assuming an average human mass to be 65 kg, we obtain $m \approx 4.4 \times 10^{11}$ kg. Then n = 44 doubling times which, at present growth rates, equates to 2770 years. Again, students are surprised at the nearness in time of such a condition and leads them to speculate on its gravitational consequences.

III. RESOURCE CONSUMPTION

The consumption rate a for non-renewable resource such as oil has been modeled as a relation of the form of Eqn (1) in which the dependent variable is expressed in barrels per year. To appreciate the role of the exponential rate of increase and of the significance of T_2 we pose the following questions:

Question (3): Show that under conditions of Eqn (1), in any doubling interval the amount consumed exceeds that consumed in all of history up to the start of that interval. This statement is always greeted with disbelief but is easily proved. One needs merely evaluate two integrals of Eqn (1), one from time $t = 0$ to some future time $t = t_1$ and the other from that time to $t = t_1 + T_2$.

Question (4): Compute the lifetime of an oil supply undergoing an exponentially increasing rate of consumptions. Again we perform an integration of Eqn (1) with the unknown being the upper limit of integration. This is equated to the size of the resource for which we take an extreme case: assume Earth to be one giant oil tank $\approx 10^{21}$ bbls. If use k = 7%/yr, a value which persisted for most of the 20 Century, the lifetime is ~300 years. At this point a student will exclaim that we don't possess that much oil. **Indeed!!**

REFERENCES

[1]. D. W. Kraft, http://asee-ne.org/proceedings/2014/index.htm.

[2]. A. A. Bartlett, Am. J. Phys 46, 887 (1978).